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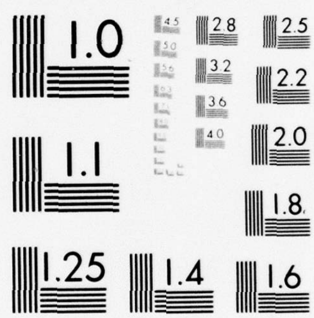

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THREE ADDITIONAL ELEMENTS FOR THE FINITE ELEMENT PROGRAM FINEL

by

John C. Adamchak

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DEPARTMENT OF STRUCTURAL MECHANICS

RESEARCH AND DEVELOPMENT REPORT

December 1970

Report 3525

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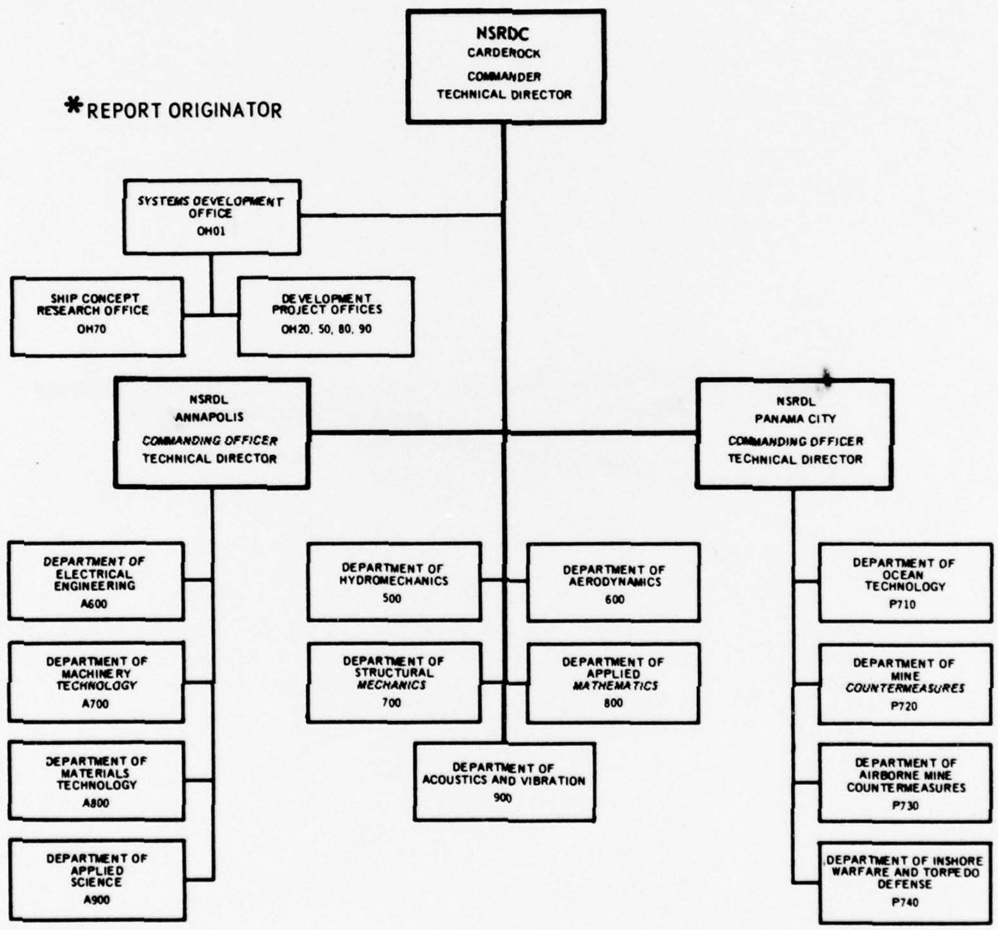

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NOTATION

A	Area of triangular element
a	x_j, "length" of rectangular element
$\underset{\sim}{B}, \underset{\sim}{J}$	Matrices employed in the calculation of $\underset{\sim}{k}$ for the rectangular membrane element
b	y_k, "width" of rectangular element
b_{ij}	k_{ij} of the bending element stiffness matrix
C_d	Deflection coefficient for plate bending examples
$C_{11}, C_{12}, C_{21}, C_{22}, C_{31}, C_{32}$	Constants used in the calculation of the stiffness matrix for the rectangular membrane element.
D	Plate flexural rigidity
d	x_j, "*base*" *of triangular element*
E	Young's modulus
e	x_k, x coordinate of node k of the triangular element
$\underset{\sim}{F}, \underset{\sim}{L}, \underset{\sim}{S}$	Matrices used in the calculation of the stiffness matrix for the triangular bending element
f	y_k, "height" of the triangular element
K,M,N	Node moment components about element x, y, z axes, respectively
$\underset{\sim}{k}$	Element stiffness matrix (2nd definition)
$\underset{\sim}{k}^*$	Element stiffness matrix of "pinned" triangular bending element
$\underset{\sim}{k}_a, \underset{\sim}{k}_b, \underset{\sim}{k}_c, \underset{\sim}{k}_d$	Matrices used in the calculation of the stiffness matrix for the rectangular bending element

$\underset{\sim}{k}$	Element stiffness matrix (1st definition)
m	Total number of nodal displacement components of an element
m_{ij}	k_{ij} of the membrane element stiffness matrix
n	Total number of rigid body degrees of freedom of an element
$\underset{\sim}{P}$	Matrix relating nodal displacements to element deformations
Q	Plate concentrated lateral force
$\underset{\sim}{R}, \underset{\sim}{R}^{(1)}, \underset{\sim}{R}^{(2)}$	Nodal force matrices
r	Parameter defining plate mesh size
s	Edge dimension of square plate
$\underset{\sim}{T}$	Transformation matrix for triangular bending element
t	Uniform plate thickness
u,v,w	Node translational displacement components along element x, y, z axes, respectively
$\underset{\sim}{u}, \underset{\sim}{u}^{(1)}, \underset{\sim}{u}^{(2)}$	Nodal displacement matrices
$\underset{\sim}{v}$	Element deformation matrix
X,Y,Z	Node force components along element x, y, z axes, respectively
x_j, x_k	x coordinates of nodes j and k, respectively
y_k	y coordinate of node k
α, β, γ	Node rotational displacement components about element x, y, z axes, respectively
ν	Poisson's ratio
σ, τ	Normal, shear stress, respectively

ABSTRACT

Three additional elements have been added to the element library of the FINEL finite element program: a rectangular membrane element, a rectangular combined membrane and bending element, and a triangular combined membrane and bending element. This report describes the characteristics of the three new elements and includes data on the bending convergence behavior of the combined elements. The stiffness matrices of the three elements are presented in the appendix.

ADMINISTRATIVE INFORMATION

This project was conducted at the Naval Ship Research and Development Center as part of a study on computer methods applied to naval ship structures. Funding was provided through NAVSHIPS under Subproject SF 35 422 302, Task 11989.

INTRODUCTION

In view of the popularity of the FINEL finite element program,[1,2] an effort was made to increase its versatility and efficiency by adding several elements to the finite element

[1] References are listed on page 39 .

library. Three types of elements were chosen: a rectangular membrane element, a rectangular combined membrane and bending element, and a triangular combined membrane and bending element. The combined membrane and bending elements were selected because the original FINEL program contained no plate bending elements and thus could not be applied to a large class of problems that are both interesting and significant. The rectangular membrane element was added to the element library because of its potential for increasing the operating efficiency of the program. The original program already included a general quadrilateral membrane element, but past experience with FINEL had indicated that this quadrilateral element was most frequently used with rectangular geometry. Thus, the internal mathematics involved justified inclusion of a rectangular membrane element as a separate element in the library.

The most important characteristics of the specific three new elements are described herein.

STIFFNESS MATRIX DEFINITIONS

Before describing the specific elements, it is necessary to discuss briefly two somewhat different, but entirely consistent, definitions of the element stiffness matrix because reference will be made later to both of these definitions. An understanding of basic finite element theory is assumed here; some readers new to this technique may wish to first consult

one of the more comprehensive sources available on the theory.[3,4]

The local element coordinate system and the sign convention employed in this report and in the FINEL program are illustrated in Figure 1. A right-handed system is adopted here. Translational displacement and force components are positive in the positive direction of the corresponding element axes whereas rotational displacement and moment components are positive in the usual right-handed sense. The coordinate system and sign convention differ in certain respects from those employed in various references cited herein. In view of all the ramifications which arise because of these differences, extreme care when consulting these (or any) references on finite element theory cannot be overemphasized.

Consider some general finite element that possesses a total of m nodal displacement components and n rigid body degrees of freedom. Since the node displacements, in general, consist of the superposition of a rigid body displacement of the entire element and the deformation of the element, it is convenient to divide the complete column matrix of node displacements $\underset{\sim}{u}$ into two smaller column matrices

$$\underset{\sim}{u} = \begin{Bmatrix} \underset{\sim}{u}^{(1)} \\ \underset{\sim}{u}^{(2)} \end{Bmatrix} \tag{1}$$

where $\underset{\sim}{u}^{(2)}$ is a matrix of n node displacements properly chosen to express the rigid body displacement of the element, and

$\underset{\sim}{u}^{(1)}$ is the matrix of the remaining m-n displacements. The complete set of node forces represented by the matrix $\underset{\sim}{R}$ may be similarly divided into two smaller matrices $\underset{\sim}{R}^{(1)}$ and $\underset{\sim}{R}^{(2)}$

$$\underset{\sim}{R} = \begin{Bmatrix} \underset{\sim}{R}^{(1)} \\ \underset{\sim}{R}^{(2)} \end{Bmatrix} \tag{2}$$

where the components of $\underset{\sim}{R}^{(1)}$ and $\underset{\sim}{R}^{(2)}$ are in one-to-one correspondence with the displacement components of $\underset{\sim}{u}^{(1)}$ and $\underset{\sim}{u}^{(2)}$, respectively. Reference 2 defines an element <u>deformation</u> matrix $\underset{\sim}{v}$ which is equal to the difference between the total $\underset{\sim}{u}^{(1)}$ and that part of $\underset{\sim}{u}^{(1)}$ resulting from the rigid body displacement (represented in terms of the matrix $\underset{\sim}{u}^{(2)}$). Thus the deformation matrix $\underset{\sim}{v}$ can be related to the node displacements by means of a matrix $\underset{\sim}{P}$ such that

$$\begin{aligned} \underset{\sim}{v} &= \underset{\sim}{P} \begin{Bmatrix} \underset{\sim}{u}^{(1)} \\ \underset{\sim}{u}^{(2)} \end{Bmatrix} \\ &= \underset{\sim}{P} \cdot \underset{\sim}{u} \end{aligned} \tag{3}$$

Using the matrix defining the element deformation, the stiffness matrix $\underset{\sim}{k}$ is defined according to the matrix equation

$$\underset{\sim}{R}^{(1)} = \underset{\sim}{k} \cdot \underset{\sim}{v} \tag{4}$$

This first definition of a stiffness matrix is the one employed by the FINEL program. This is not, however, the definition most often encountered in the literature on finite element theory. Nevertheless, as will be demonstrated, the

second definition can be derived in a straightforward manner from Equation (4).

It can be easily shown that the nodal force matrices $\underset{\sim}{R}$ and $\underset{\sim}{R}^{(1)}$ are related by the matrix $\underset{\sim}{P}$ according to the expression

$$\underset{\sim}{R} = \underset{\sim}{P}^T \cdot \underset{\sim}{R}^{(1)} \tag{5}$$

where the superscript T indicates the matrix transpose. If Equation (3) for $\underset{\sim}{v}$ is substituted into Equation (4) and both sides of the resulting expression are premultiplied by $\underset{\sim}{P}^T$, the following relationship is obtained:

$$\underset{\sim}{P}^T \underset{\sim}{R}^{(1)} = \underset{\sim}{P}^T \cdot \underset{\sim}{\mathcal{k}} \cdot \underset{\sim}{P} \cdot \underset{\sim}{u} \tag{6a}$$

Using Equation (5), the above may be rewritten as

$$\underset{\sim}{R} = \underset{\sim}{P}^T \cdot \underset{\sim}{\mathcal{k}} \cdot \underset{\sim}{P} \cdot \underset{\sim}{u} \tag{6b}$$

By combining terms, this expression may be further rewritten as

$$\underset{\sim}{R} = \underset{\sim}{k} \cdot \underset{\sim}{u} \tag{7}$$

where the matrix $\underset{\sim}{k}$ is defined as

$$\underset{\sim}{k} = \underset{\sim}{P}^T \cdot \underset{\sim}{\mathcal{k}} \cdot \underset{\sim}{P} \tag{8}$$

It is now easily seen that the matrix $\underset{\sim}{k}$ defined by Equation (7) also represents a stiffness matrix.* This second definition of the stiffness matrix is the one which appears

---

* The matrix $\underset{\sim}{k}$ defined here should not be confused with the similar symbol used in Reference 2 since in this case their definitions differ.

more frequently in the literature on finite element theory. The simple relationship between the two stiffness matrices $\underset{\sim}{\mathscr{k}}$ and $\underset{\sim}{k}$ is given by Equation (8). This expression indicates how the matrix $\underset{\sim}{k}$ may be calculated from a knowledge of $\underset{\sim}{\mathscr{k}}$ (and, obviously, $\underset{\sim}{P}$), but it can also be shown that $\underset{\sim}{\mathscr{k}}$ may be obtained from $\underset{\sim}{k}$ simply by inspection.

THE THREE NEW ELEMENTS

THE RECTANGULAR MEMBRANE ELEMENT

The rectangular membrane element is an isotropic element of constant thickness t with linearly varying normal stress distributions along the element axes and a constant shear stress distribution. This element and its orientation with respect to the local element coordinate system is shown in Figure 2.

This rectangular element represents a special case of the general quadrilateral membrane element included in the library of the original FINEL program (as Element 2) and described in Reference 2. The rectangular geometry results in certain mathematical simplifications which, at least in theory, produce an element whose stiffness matrix can be calculated with greater speed and accuracy than that of the general quadrilateral.

The stiffness matrices for both the general quadrilateral and the rectangular membrane element are defined in terms of node displacement and node force matrices given by

$$\underset{\sim}{u} = \{u_i v_i u_j v_j u_k v_k u_l v_l\} \tag{9a}$$

$$\underset{\sim}{u}^{(1)} = \{u_j u_k v_k u_l v_l\} \tag{9b}$$

$$\underset{\sim}{u}^{(2)} = \{u_i v_i v_j\} \tag{9c}$$

and

$$\underset{\sim}{R} = \{X_i Y_i X_j Y_j X_k Y_k X_l Y_l\} \tag{10a}$$

$$\underset{\sim}{R}^{(1)} = \{X_j X_k Y_k X_l Y_l\} \tag{10b}$$

$$\underset{\sim}{R}^{(2)} = \{X_i Y_i Y_j\} \tag{10c}$$

Employing the customary energy approach, it is shown[2] that the stiffness matrix $\underset{\sim}{k}$ is computed from the matrix product

$$\underset{\sim}{k} = (\underset{\sim}{B}^{-1})^T \cdot \underset{\sim}{J} \cdot \underset{\sim}{B}^{-1} \tag{11}$$

where the matrices $\underset{\sim}{B}$ and $\underset{\sim}{J}$ are functions of the element geometry and material properties. These matrices are presented in Reference 2 for the general quadrilateral.

For the general quadrilateral element, the form of matrix $\underset{\sim}{B}$ (a square matrix of rank 5) is sufficiently complex to make manual inversion as required by Equation (11) impractical. Consequently, the stiffness matrix for this element is calculated in the FINEL program by numerically inverting the matrix $\underset{\sim}{B}$ and performing the necessary matrix multiplications within the program.

The simplified geometry of the rectangular membrane element allows for more direct calculation of the stiffness matrix. The matrix $\underset{\sim}{B}$ in this case is quite suitable for manual inversion and thus it is possible to perform the necessary matrix multiplications algebraically to determine in closed form the individual components of the stiffness matrix.

Matrices $\underset{\sim}{B}$, $\underset{\sim}{B}^{-1}$, $\underset{\sim}{J}$, $\underset{\sim}{k}$, $\underset{\sim}{P}$, and $\underset{\sim}{k}$ (in symbolic form) for the rectangular membrane element are shown in the Appendix to this report.

THE RECTANGULAR COMBINED MEMBRANE AND BENDING ELEMENT

Like the pure membrane element, the rectangular combined membrane and bending element is an isotropic element with uniform thickness t. The stiffness matrix for this element is obtained by combining the stiffness matrices of a rectangular membrane element and a rectangular bending element. Since this indicates that the membrane and the bending behavior are assumed to be uncoupled, the two forms of behavior may be considered separately. The Appendix describes the manner in which the stiffness matrices corresponding to the two forms of behavior are combined to form the combined stiffness matrix.

The rectangular membrane element chosen to provide the membrane stiffness matrix is the element discussed in the immediately preceding section. Consequently the membrane characteristics of this element and those of the combined membrane and bending element are identical. Since the

characteristics of this membrane element have already been adequately discussed, the remainder of this section will be devoted mainly to a description of the rectangular bending element selected to provide the bending stiffness matrix.

For the combined membrane and bending element, the stiffness matrices $\underset{\sim}{k}$ and $\underset{\sim}{\ell}$ are defined in terms of the node displacement and node force matrices given by

$$\underset{\sim}{u} = \{u_i v_i w_i \alpha_i \beta_i u_j v_j w_j \alpha_j \beta_j u_k v_k w_k \alpha_k \beta_k u_l v_l w_l \alpha_l \beta_l\} \quad (12a)$$

$$\underset{\sim}{u}^{(1)} = \{u_j w_j \alpha_j \beta_j u_k v_k w_k \alpha_k \beta_k u_l v_l w_l \alpha_l \beta_l\} \quad (12b)$$

$$\underset{\sim}{u}^{(2)} = \{u_i v_i w_i \alpha_i \beta_i v_j\} \quad (12c)$$

and

$$\underset{\sim}{R} = \{X_i Y_i Z_i K_i M_i X_j Y_j Z_j K_j M_j X_k Y_k Z_k K_k M_k X_l Y_l Z_l K_l M_l\} \quad (13a)$$

$$\underset{\sim}{R}^{(1)} = \{X_j Z_j K_j M_j X_k Y_k Z_k K_k M_k X_l Y_l Z_l K_l M_l\} \quad (13b)$$

$$\underset{\sim}{R}^{(2)} = \{X_i Y_i Z_i K_i M_i Y_j\} \quad (13c)$$

In these expressions, the two rotations α and β are related to the lateral deflection w by the derivatives

$$\alpha = \frac{\partial w}{\partial y} \quad (14)$$

$$\beta = -\frac{\partial w}{\partial x} \quad (15)$$

For the pure bending element, the matrices $\underset{\sim}{u}$ and $\underset{\sim}{R}$ become

$$\underset{\sim}{u} = \{w_i \alpha_i \beta_i w_j \alpha_j \beta_j w_k \alpha_k \beta_k w_l \alpha_l \beta_l\} \quad (16)$$

$$\underset{\sim}{R} = \{Z_i K_i M_i Z_j K_j M_j Z_k K_k M_k Z_l K_l M_l\} \quad (17)$$

The rectangular bending element chosen is described in References 5 and 6 and referred to in the latter reference as the Melosh or M element. In bending, this element is assumed to distort along the edges with shapes defined by beam displacement functions. These displacements are assumed to decrease linearly toward the opposite edge. In addition it is assumed that twisting effects are resisted by a state of uniform twist in the element. This state distorts the element to a simple hyperbolic-paraboloid shape. Plots documenting the excellent convergence characteristics of this element are presented in Reference 6 as well as in the following section of this report. These latter plots were constructed from new data obtained with the use of the FINEL program.

The Appendix shows the stiffness matrix $\underset{\sim}{k}$ for this rectangular bending element as well as the matrices $\underset{\sim}{k}$ and $\underset{\sim}{k}$ (in symbolic form) for the combined membrane and bending element (oriented with respect to the element coordinate system as shown in Figure 2).

THE TRIANGULAR COMBINED MEMBRANE AND BENDING ELEMENT

As with the rectangular combined element, the stiffness matrix for the triangular combined membrane and bending element is obtained by combining the stiffness matrices of a triangular membrane element and a triangular bending element. The membrane element employed is the well known "constant stress" triangle included in the original FINEL program as

Element 1 and described in Reference 2. The assumed membrane stress distribution for this element is illustrated in Figure 3. The triangular bending element chosen to provide the bending stiffness matrix is described in the following paragraphs.

For the triangular combined element, oriented as shown in Figure 3, the stiffness matrices $\underset{\sim}{\ell}$ and $\underset{\sim}{k}$ are defined in terms of the node displacement and node force matrices represented by

$$\underset{\sim}{u} = \{u_i v_i w_i \alpha_i \beta_i u_j v_j w_j \alpha_j \beta_j u_\ell v_\ell w_\ell \alpha_\ell \beta_\ell\} \tag{18a}$$

$$\underset{\sim}{u}^{(1)} = \{u_j w_j \alpha_j \beta_j u_\ell v_\ell w_\ell \alpha_\ell \beta_\ell\} \tag{18b}$$

$$\underset{\sim}{u}^{(2)} = \{u_i v_i w_i \alpha_i \beta_i v_j\} \tag{18c}$$

and

$$\underset{\sim}{R} = \{X_i Y_i Z_i K_i M_i X_j Y_j Z_j K_j M_j X_\ell Y_\ell Z_\ell K_\ell M_\ell\} \tag{19a}$$

$$\underset{\sim}{R}^{(1)} = \{X_j Z_j K_j M_j X_\ell Y_\ell Z_\ell K_\ell M_\ell\} \tag{19b}$$

$$\underset{\sim}{R}^{(2)} = \{X_i Y_i Z_i K_i M_i Y_j\} \tag{19c}$$

For the pure bending element, the $\underset{\sim}{u}$ and $\underset{\sim}{R}$ matrices become

$$\underset{\sim}{u} = \{w_i \alpha_i \beta_i w_j \alpha_j \beta_j w_\ell \alpha_\ell \beta_\ell\} \tag{20}$$

$$\underset{\sim}{R} = \{Z_i K_i M_i Z_j K_j M_j Z_\ell K_\ell M_\ell\} \tag{21}$$

The particular triangular bending element selected is a nonconforming element which is described in detail in References 3 and 7. The displacement function for this element uses the so-called "area coordinates" to ensure symmetry and

simplicity of statement. The formulation of this function is based on the hypothetically necessary condition (for convergence) that the displacement function is capable of representing constant curvature (strain) states throughout the finite element, irrespective of element size or shape.

The actual derivation of the stiffness matrix $\underset{\sim}{k}$ for this element is carried out by first determining a stiffness matrix, labeled $\underset{\sim}{k}^{*}$, for a simply supported (at the nodes) element, and then transforming this matrix to the desired stiffness matrix. This transformation is carried out by means of a transformation matrix $\underset{\sim}{T}$ which is a function of the element geometry and its orientation with respect to the local element coordinate system. The expression relating $\underset{\sim}{k}^{*}$, $\underset{\sim}{T}$, and $\underset{\sim}{k}$ is

$$\underset{\sim}{k} = \underset{\sim}{T}^{T} \cdot \underset{\sim}{k}^{*} \cdot \underset{\sim}{T} \tag{22}$$

Because of the complexity of the components of the $\underset{\sim}{k}^{*}$ matrix, the matrix $\underset{\sim}{k}$ is computed numerically in the FINEL program by performing the matrix multiplications indicated by Equation (22).

The Appendix contains the $\underset{\sim}{k}^{*}$ and $\underset{\sim}{T}$ matrices for the triangular bending element oriented as shown in Figure 3 as well as the matrices $\underset{\sim}{k}$ and $\underset{\sim}{\mathit{k}}$ (in symbolic form) for the triangular combined element.

BENDING CONVERGENCE CHARACTERISTICS OF THE COMBINED ELEMENTS

In order to evaluate the bending behavior of the two combined elements, a series of analyses was carried out on a uniform square plate laterally loaded by a concentrated force at its center. Two edge support conditions were considered: all edges simply supported, and all edges clamped. This problem was chosen because of the availability of the "exact" solutions in a number of texts.

In this study, the deflection of the center of the plate w_c was taken as the measure of the quality of the approximation. This deflection is represented by a nondimensional "deflection coefficient," designated C_d, which is defined as

$$C_d = \frac{w_c D}{Q s^2} \tag{23}$$

where D is the flexural rigidity of the plate,

s is the edge dimension of the plate, and

Q is the magnitude of the concentrated force.

Using linear plate bending theory, the "exact" solutions for the deflection coefficient C_d for the two cases of edge support are found to be

$$C_d = 0.01160 \quad \text{Simply Supported}$$

and

$$C_d = 0.00560 \quad \text{Clamped}$$

Because of its double symmetry, only one-quarter of the plate was considered in the finite element analysis. The convergence characteristics of the two elements in bending were studied by using a series of different element mesh arrangements and sizes in the analysis of each case. The mesh size is indicated by a mesh number r which refers to the number of rectangles along the side of the complete plate. The three basic mesh arrangements employed in this study are shown in Figure 4 for a value of the mesh number r = 8. Figure 5 shows another mesh arrangement, in this case for r = 4, for which two additional data points were plotted. This mesh was investigated primarily to allow for additional comparisons of data produced using the FINEL program with that published in Reference 7. As is noted in that reference, however, this mesh arrangement does not converge to the correct solution when used with nonconforming elements.

The results of this series of analyses are presented in Figures 6 and 7. These figures are plots of the nondimensional deflection coefficient C_d versus the mesh size r.

Both Figures 6 and 7 illustrate the excellent bending convergence characteristics of the rectangular combined element. Other known results obtained with this element in its pure bending form also demonstrate excellent convergence.[6] Comparisons with these older data suggest that the drop in the curve for Mesh A at r = 16 on Figure 6 is the result of program round-off error. These examples indicate that the use of

this rectangular element can give good results for plate bending problems even with relatively coarse meshes.

It is also apparent from the figures that good results for plate bending problems are possible using the triangular combined element, but the rate of convergence is very much a function of the mesh arrangement. Discussions of convergence problems with nonconforming triangular elements indicate that certain mesh arrangements do not converge to the correct solution.[3,7] For the arrangements considered here, these sources indicate that Meshes B and C will converge to the correct solution whereas Mesh D will not. Readers are urged to examine the references cited for a more detailed discussion of this subject. This behavior suggests that for best results with plate bending problems, rectangular elements should be used whenever feasible. Should the geometry of the problem be such that the use of triangular elements is either desired or unavoidable, it is essential that the user be aware of the convergence problems associated with their use.

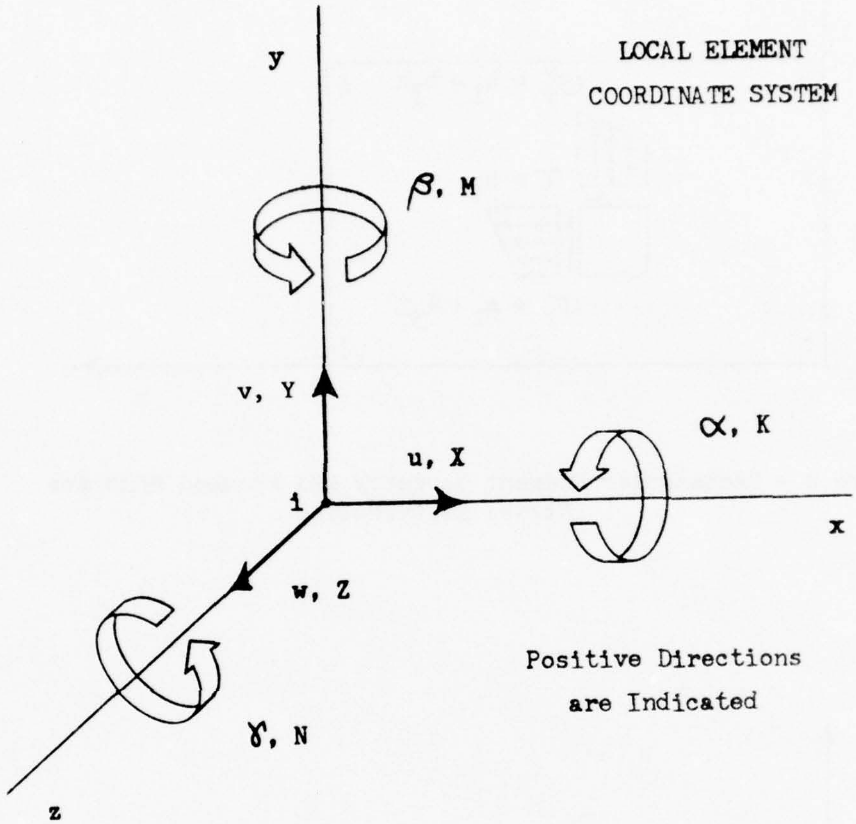
SUMMARY

Three new elements have been added to the library of the FINEL finite element program: a rectangular membrane element, a rectangular combined membrane and bending element, and a triangular combined membrane and bending element. The rectangular membrane element is a special case of the general

quadrilateral membrane element included in the original FINEL program.

The rectangular combined element exhibits excellent convergence characteristics in bending. Good results for plate bending problems can be achieved with this element, even for relatively coarse meshes. This element is recommended for such problems whenever the problem geometry permits.

Good results for plate bending problems are also possible with the triangular combined element, but the rate of convergence is very much a function of mesh arrangement. Certain mesh patterns do not converge to the correct solution. An understanding of the convergence problems with nonconforming bending triangles is necessary for users of this element.



DISPLACEMENT COMPONENTS

Translational - u, v, w

Rotational - α, β, γ

LOAD COMPONENTS

Force - X, Y, Z

Moment - K, M, N

Figure 1 - Element Coordinate System

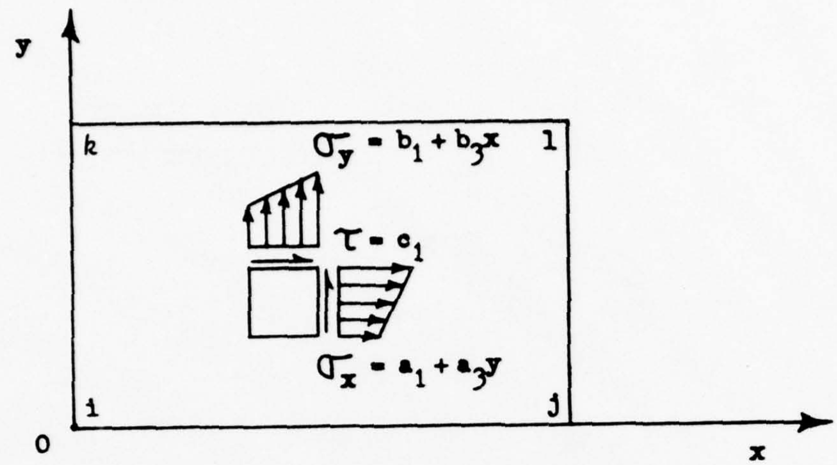


Figure 2 - Rectangular Element Geometry and Assumed Membrane Stress Distribution

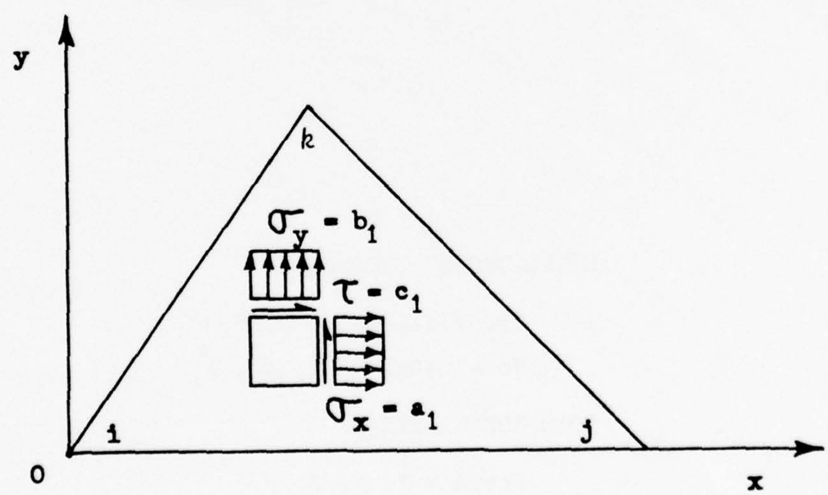
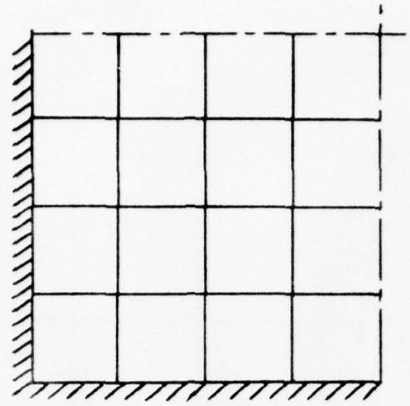
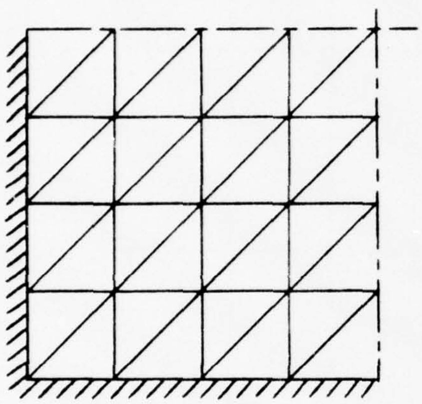


Figure 3 - Triangular Element Geometry and Assumed Membrane Stress Distribution



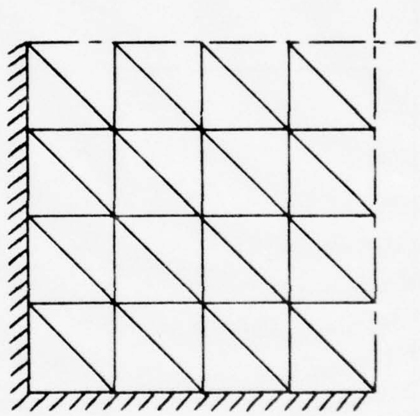
MESH A

Rectangular Elements



MESH B

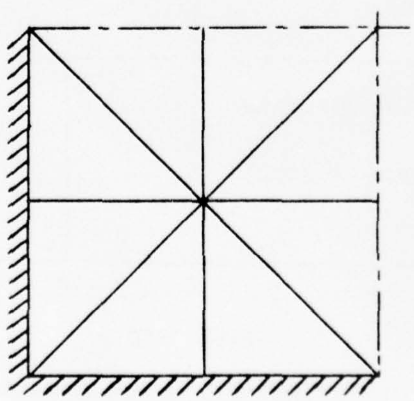
Triangular Elements



MESH C

Triangular Elements

Figure 4 - Basic Mesh Arrangements (r = 8)



MESH D

Triangular Elements

Figure 5 - Special Mesh D (r = 4)

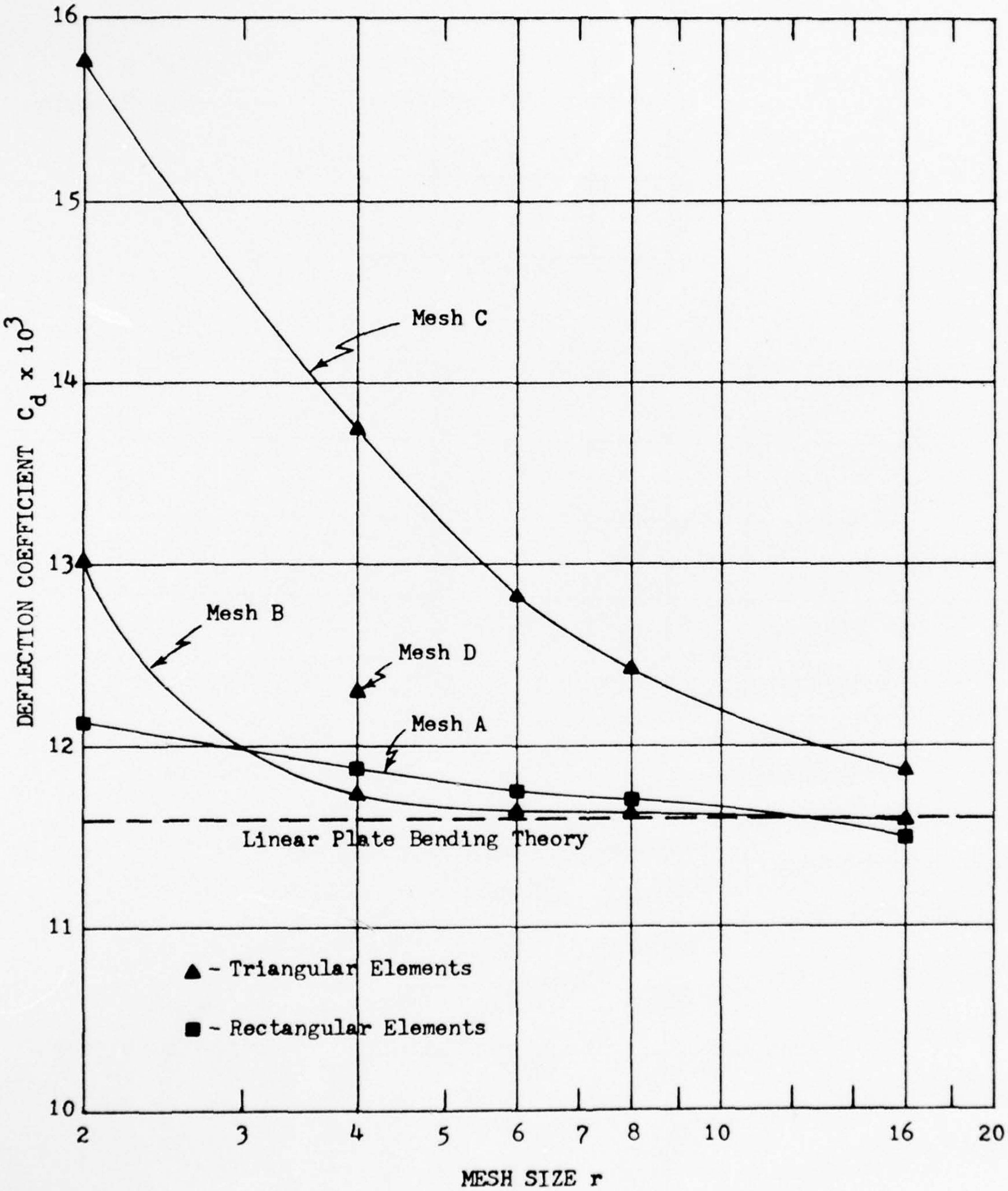


Figure 6 - Deflection Coefficient versus Mesh Size for Simply Supported Plate

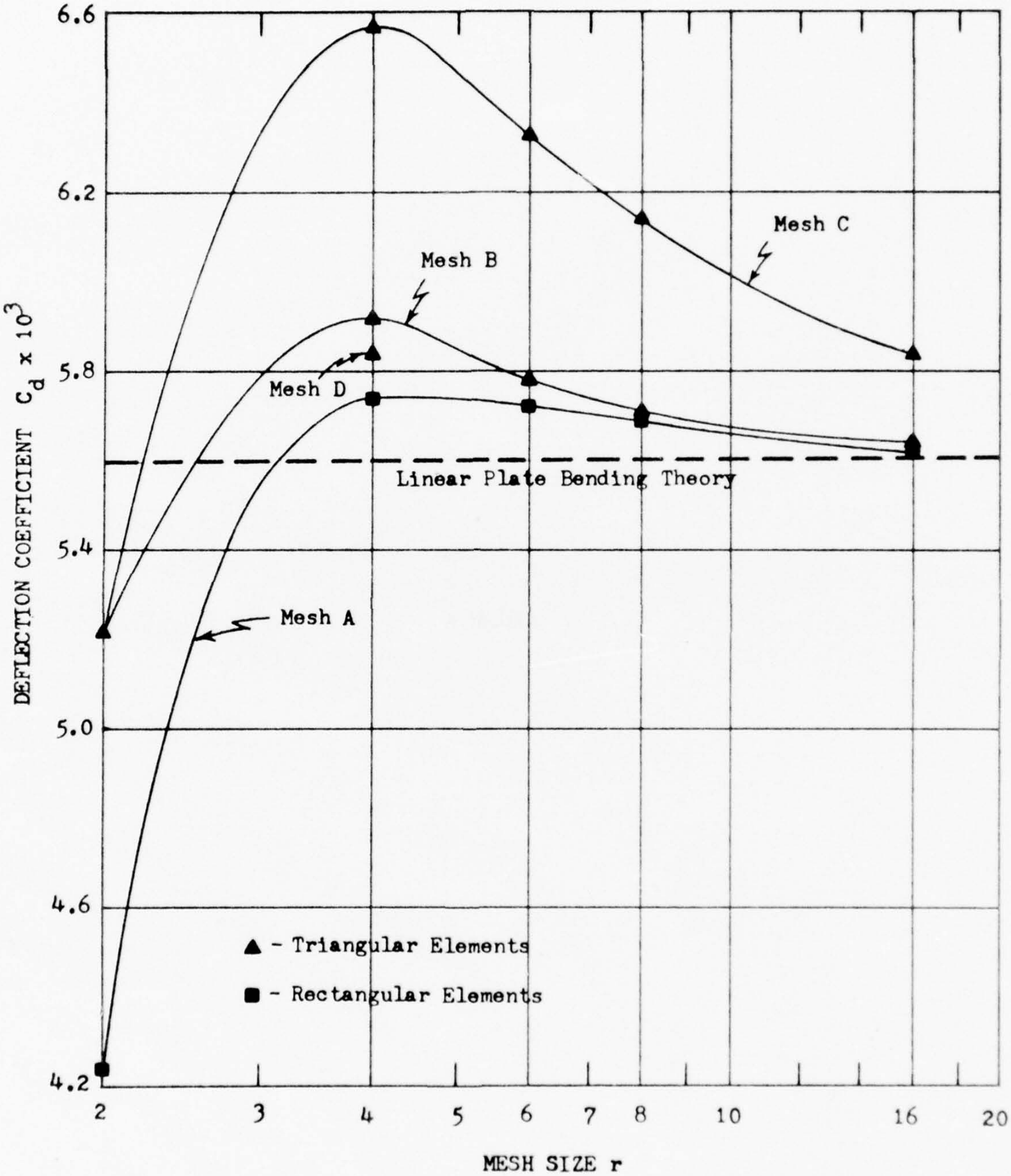


Figure 7 - Deflection Coefficient versus Mesh Size for Clamped Plate

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APPENDIX

ELEMENT STIFFNESS MATRICES

RECTANGULAR MEMBRANE ELEMENT

The stiffness matrix $\underset{\sim}{k}$ of the rectangular membrane element is computed from a matrix product given by

$$\underset{\sim}{k} = (\underset{\sim}{B}^{-1})^{T} \cdot \underset{\sim}{J} \cdot \underset{\sim}{B}^{-1} \tag{11}$$

For this rectangular element, Matrices $\underset{\sim}{B}$, $\underset{\sim}{B}^{-1}$, and $\underset{\sim}{J}$ have the following forms:

$$\underset{\sim}{B} = \frac{1}{E}\begin{bmatrix} x_j & 0 & -\nu x_j & -\frac{1}{2}\nu x_j^2 & 0 \\ 0 & -\frac{1}{2}x_j x_k & 0 & -\frac{1}{2}y_k^2 & 2(1+\nu)y_k \\ -\nu y_k & -\frac{1}{2}\nu y_k^2 & y_k & 0 & 0 \\ x_j & \frac{1}{2}x_j y_k & -\nu x_j & -\frac{1}{2}(\nu x_j^2 + y_k^2) & 2(1+\nu)y_k \\ -\nu y_k & -\frac{1}{2}\nu y_k^2 & y_k & x_j y_k & 0 \end{bmatrix}$$

$$
\underset{\sim}{B}^{-1} = \frac{E}{2(1-\nu^2)}
\begin{bmatrix}
\frac{2-\nu^2}{x_j} & -\frac{\nu^2}{x_j} & \frac{\nu}{y_k} & \frac{\nu^2}{x_j} & \frac{\nu}{y_k} \\
-\frac{2(1-\nu^2)}{x_j y_k} & -\frac{2(1-\nu^2)}{x_j y_k} & 0 & \frac{2(1-\nu^2)}{x_j y_k} & 0 \\
\frac{\nu}{x_j} & -\frac{\nu}{x_j} & \frac{2-\nu^2}{y_k} & \frac{\nu}{x_j} & \frac{\nu^2}{y_k} \\
0 & 0 & -\frac{2(1-\nu^2)}{x_j y_k} & 0 & \frac{2(1-\nu^2)}{x_j y_k} \\
-\frac{1-\nu}{2y_k} & \frac{1-\nu}{2y_k} & -\frac{1-\nu}{2x_j} & \frac{1-\nu}{2y_k} & \frac{1-\nu}{2x_j}
\end{bmatrix}
$$

$$
\underset{\sim}{J} = \frac{t}{E}
\begin{bmatrix}
x_j y_k & & & & \\
\frac{1}{2} x_j y_k^2 & \frac{1}{3} x_j y_k^3 & & \text{SYMMETRIC} & \\
-\nu x_j y_k & -\frac{1}{2}\nu x_j y_k^2 & x_j y_k & & \\
-\frac{\nu}{2} x_j^2 y_k & -\frac{\nu}{4} x_j^2 y_k^2 & \frac{1}{2} x_j^2 y_k & \frac{1}{3} x_j^3 y_k & \\
0 & 0 & 0 & 0 & 2(1+\nu) x_j y_k
\end{bmatrix}
$$

If the matrix multiplications indicated by Equation (11) are performed, the stiffness matrix $\underset{\sim}{k}$ is the result:

$$\underset{\sim}{k} = \frac{Et}{4(1-\nu^2)} \begin{bmatrix} (C_{11}+C_{32}) & & & & \\ (-C_{21}-C_{32}) & (C_{11}+C_{32}) & & \text{SYMMETRIC} & \\ \frac{1}{2}(1+\nu) & -\frac{1}{2}(1+\nu) & (C_{12}+C_{31}) & & \\ (C_{21}-C_{32}) & (-C_{11}+C_{32}) & -\frac{1}{2}(1-3\nu) & (C_{11}+C_{32}) & \\ -\frac{1}{2}(1-3\nu) & \frac{1}{2}(1-3\nu) & (C_{22}-C_{31}) & \frac{1}{2}(1+\nu) & (C_{12}+C_{31}) \end{bmatrix}$$

where $C_{11} = (y_k/x_j)[1+\frac{1}{3}(1-\nu^2)]$

$C_{12} = (x_j/y_k)[1+\frac{1}{3}(1-\nu^2)]$

$C_{21} = (y_k/x_j)[1-\frac{1}{3}(1-\nu^2)]$

$C_{22} = (x_j/y_k)[1-\frac{1}{3}(1-\nu^2)]$

$C_{31} = \frac{1}{2}(y_k/x_j)(1-\nu)$

$C_{32} = \frac{1}{2}(x_j/y_k)(1-\nu)$

Matrix $\underset{\sim}{P}$ is required in order to obtain the stiffness matrix $\underset{\sim}{k}$ from $\underset{\sim}{\mathcal{k}}$ as indicated by Equation (8). This matrix is given by

$$\underset{\sim}{P} = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -y_{\mathcal{k}}/x_j & 0 & y_{\mathcal{k}}/x_j & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & -y_{\mathcal{k}}/x_j & 0 & y_{\mathcal{k}}/x_j & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Performing the operations indicated by Equation (8) will produce the matrix $\underset{\sim}{k}$. This matrix may be written in symbolic form as

$$\underset{\sim}{k} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} & k_{17} & k_{18} \\ & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} & k_{27} & k_{28} \\ & & \mathcal{k}_{11} & k_{34} & \mathcal{k}_{12} & \mathcal{k}_{13} & \mathcal{k}_{14} & \mathcal{k}_{15} \\ & & & k_{44} & k_{45} & k_{46} & k_{47} & k_{48} \\ & & & & \mathcal{k}_{22} & \mathcal{k}_{23} & \mathcal{k}_{24} & \mathcal{k}_{25} \\ & \text{SYMMETRIC} & & & & \mathcal{k}_{33} & \mathcal{k}_{34} & \mathcal{k}_{35} \\ & & & & & & \mathcal{k}_{44} & \mathcal{k}_{45} \\ & & & & & & & \mathcal{k}_{55} \end{bmatrix}$$

in which it may be noted the entire $\underset{\sim}{\mathcal{k}}$ matrix appears. This demonstrates how the $\underset{\sim}{\mathcal{k}}$ stiffness matrix can be determined

from the $\underset{\sim}{k}$ stiffness matrix simply by inspection when $\underset{\sim}{u}^{(1)}$ and $\underset{\sim}{u}^{(2)}$ are known.

RECTANGULAR BENDING ELEMENT

The stiffness matrix $\underset{\sim}{k}$ for the rectangular bending element is defined in terms of the $\underset{\sim}{u}$ and $\underset{\sim}{R}$ matrices described earlier as Equations (16) and (17). This matrix can be more conveniently written as the sum of four considerably simpler matrices

$$\underset{\sim}{k} = \frac{D}{ab} \{\underset{\sim}{k}_a + \underset{\sim}{k}_b + \underset{\sim}{k}_c + \underset{\sim}{k}_d\}$$

where $a = x_j$

$b = y_k$

$$D = \frac{Et^3}{12(1-\nu^2)}$$

The matrices $\underset{\sim}{k}_a$, $\underset{\sim}{k}_b$, $\underset{\sim}{k}_c$, and $\underset{\sim}{k}_d$ are the following

$$\underset{\sim}{k}_a = \left(\frac{b}{a}\right)^2 \begin{bmatrix} 6 & & & & & & & & & & & \\ 0 & 0 & & & & & & & & & & \\ -3a & 0 & 2a^2 & & & & & & \text{SYMMETRIC} & & & \\ -6 & 0 & 3a & 6 & & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & & & & & & & \\ -3a & 0 & a^2 & 3a & 0 & 2a^2 & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & -3a & 0 & 2a^2 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & -6 & 0 & 3a & 6 & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & -3a & 0 & a^2 & 3a & 0 & 2a^2 \end{bmatrix}$$

$$
\underset{\sim}{k}_b = \frac{a^2}{b^2}
\begin{bmatrix}
6 & & & & & & & & & & & \\
3b & 2 & & & & & & & & & & \\
0 & 0 & 0 & & & & & & & \text{SYMMETRIC} & & \\
0 & 0 & 0 & 6 & & & & & & & & \\
0 & 0 & 0 & 3b & 2b^2 & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & & & & & & \\
-6 & -3b & 0 & 0 & 0 & 0 & 6 & & & & & \\
3b & b^2 & 0 & 0 & 0 & 0 & -3b & 2b^2 & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
0 & 0 & 0 & -6 & -3b & 0 & 0 & 0 & 0 & 6 & & \\
0 & 0 & 0 & 3b & b^2 & 0 & 0 & 0 & 0 & -3b & 2b^2 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\underset{\sim}{k}_c = \frac{\nu}{16}
\begin{bmatrix}
72 & & & & & & & & & & & \\
30b & 0 & & & & & & & & & & \\
-30a & -25ab & 0 & & & & & & \text{SYMMETRIC} & & & \\
-72 & -30b & 6a & 72 & & & & & & & & \\
-30b & 0 & 5ab & 30b & 0 & & & & & & & \\
-6a & -5ab & 0 & 30a & 25ab & 0 & & & & & & \\
-72 & -6b & 30a & 72 & 6b & 6a & 72 & & & & & \\
6b & 0 & -5ab & -6b & 0 & -ab & -30b & 0 & & & & \\
30a & 5ab & 0 & -6a & -ab & 0 & -30a & 25ab & 0 & & & \\
72 & 6b & -6a & -72 & -6b & -30a & -72 & 30b & 6a & 72 & & \\
-6b & 0 & ab & 6b & 0 & 5ab & 30b & 0 & -5ab & -30b & 0 & \\
6a & ab & 0 & -30a & -5ab & 0 & -6a & 5ab & 0 & 30a & -25ab & 0
\end{bmatrix}
$$

$$
\underset{\sim}{k}_d = 2(1-\nu)\begin{bmatrix}
1 & & & & & & & & & & & \\
0 & 0 & & & & & & & & & & \\
0 & 0 & 0 & & & & & & & & \text{SYMMETRIC} & \\
-1 & 0 & 0 & 1 & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & & & & & & \\
-1 & 0 & 0 & 1 & 0 & 0 & 1 & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

TRIANGULAR BENDING ELEMENT

It was stated earlier that the stiffness matrix $\underset{\sim}{k}$ of the triangular bending element, defined in terms of the $\underset{\sim}{u}$ and $\underset{\sim}{R}$ matrices presented as Equations (20) and (21), is computed from the matrix product given by

$$
\underset{\sim}{k} = \underset{\sim}{T}^T \cdot \underset{\sim}{k}^* \cdot \underset{\sim}{T} \tag{22}
$$

For the orientation of the triangle as defined in Figure 3, the transformation matrix $\underset{\sim}{T}$ is the following

$$\underset{\sim}{T} = \frac{1}{2A}\begin{bmatrix} (d-e) & 2A & 0 & e & 0 & 0 & -d & 0 & 0 \\ -f & 0 & 2A & f & 0 & 0 & 0 & 0 & 0 \\ (d-e) & 0 & 0 & e & 2A & 0 & -d & 0 & 0 \\ -f & 0 & 0 & f & 0 & 2A & 0 & 0 & 0 \\ (d-e) & 0 & 0 & e & 0 & 0 & -d & 2A & 0 \\ -f & 0 & 0 & f & 0 & 0 & 0 & 0 & 2A \end{bmatrix}$$

where $A = \frac{1}{2}x_j y_k$ the area of the triangle,

$d = x_j$

$e = x_k$, and

$f = y_k$.

The stiffness matrix $\underset{\sim}{k}^*$ for the "pinned" element can be calculated by following the procedures described in References 3 and 7. The general term of this matrix can be most conveniently represented as a function of three arbitrarily named matrices: $\underset{\sim}{F}$, $\underset{\sim}{L}$, and $\underset{\sim}{S}$.

$$k^*_{ij} = D[\{F_{1i}(F_{1j}+\nu F_{2j})+F_{2i}(F_{2j}+\nu F_{1j})$$

$$+\frac{1}{2}(1-\nu)F_{3i}F_{3j}\}A+\{F_{1i}(L_{1j}+\nu L_{2j})+L_{1i}(F_{1j}+\nu F_{2j})$$

$$+F_{2i}(L_{2j}+\nu L_{1j})+L_{2i}(F_{2j}+\nu F_{1j})+\frac{1}{2}(1-\nu)(F_{3i}L_{3j}$$

$$+L_{3i}F_{3j})\}\frac{df}{6}(d+e)+\{F_{1i}(S_{1j}+\nu S_{2j})+S_{1i}(F_{1j}$$

$$+\nu F_{2j})+F_{2i}(S_{2j}+\nu S_{1j})+S_{2i}(F_{2j}+\nu F_{1j})$$

$$+\frac{1}{2}(1-\nu)(F_{3i}S_{3j}+S_{3i}F_{3j})\}\frac{df^2}{6}+\{L_{1i}(L_{1j}+\nu L_{2j})$$

$$+L_{2i}(L_{2j}+\nu L_{1j})+\frac{1}{2}(1-\nu)L_{3i}L_{3j}\}\frac{df}{12}(d^2+de+e^2)$$

$$+\{L_{1i}(S_{1j}+\nu S_{2j})+S_{1i}(L_{1j}+\nu L_{2j})+L_{2i}(S_{2j}+\nu S_{1j})$$

$$+S_{2i}(L_{2j}+\nu L_{1j})+\frac{1}{2}(1-\nu)(L_{3i}S_{3j}+S_{3i}L_{3j})\frac{df^2}{24}(d+2e)$$

$$+\{S_{1i}(S_{1j}+\nu S_{2j})+S_{2i}(S_{2j}+\nu S_{1j})$$

$$+\frac{1}{2}(1-\nu)S_{3i}S_{3j}\}\frac{df^3}{12}]$$

The matrices $\underset{\sim}{F}$, $\underset{\sim}{L}$, and $\underset{\sim}{S}$ are defined as follows:

$$\underset{\sim}{F} = \frac{1}{8A^3}\begin{bmatrix} 0 & -4d^2f^3 & 0 & -2f^3d^2 & 0 & 0 \\ d^2f^2(4d-3e) & -d^2ef(e+d) & d^2ef^2 & d^2ef(2d-3e) & 2d^2f^2(d-e) & d^2ef(2e-3d) \\ -3d^2f^3 & d^2f^2(3d-5e) & d^2f^3 & d^2f^2(2d-5e) & -2d^2f^3 & d^2f^2(2e-d) \end{bmatrix}$$

$$\underset{\sim}{L} = \frac{1}{8A^3}\begin{bmatrix} 0 & 6df^3 & 0 & 6df^3 & 0 & 0 \\ df^2(2e-3d) & df(4e^2-3de+d^2) & df^2(2e+d) & fd(4e^2-5de+2d^2) & 2df^2(2e-d) & df(d^2+4de-4e^2) \\ 2df^3 & 2df^2(5e-3d) & 2f^3d & 2f^2d(5e-2d) & 4df^3 & 2df^2(d-2e) \end{bmatrix}$$

$$\underset{\sim}{S} = \frac{1}{8A^3}\begin{bmatrix} -df^3 & df^2(3d-5e) & -f^3d & f^2d(2d-5e) & -2df^3 & df^2(2e-d) \\ 3df(d-e)(e-2d) & 3de(d^2-e^2) & -3def(e+d) & 3de(e-2d)(d-e) & 6df(de-d^2-e^2) & 3de(e-d)(2e-d) \\ 2df^2(3d-2e) & 2df(3de-4e^2-d^2) & -2df^2(2e+d) & 2fd(5de-4e^2-2d^2) & 4df^2(d-2e) & 2df(4e^2-4de-d^2) \end{bmatrix}$$

Because of the complicated algebraic form of the components of $\underset{\sim}{k}^*$, it is necessary in the FINEL program to compute the stiffness matrix $\underset{\sim}{k}$ for this element numerically within the program according to the matrix operations dictated by Equation (22).

COMBINED MEMBRANE AND BENDING ELEMENTS

As discussed earlier, the stiffness matrices for the combined membrane and bending elements are formed by combining the stiffness matrices of the appropriate membrane and bending elements. The following notation is employed to distinguish in the combined stiffness matrices between stiffness terms

associated with the membrane stiffness matrix and terms associated with the bending stiffness matrix:

$$m_{ij} = (k_{ij})_{membrane}$$

$$b_{ij} = (k_{ij})_{bending}$$

Rectangular Combined Membrane and Bending Element

Using the notation just defined, the stiffness matrix $\underset{\sim}{k}$ for the rectangular combined membrane and bending element, defined in terms of the $\underset{\sim}{u}$ and $\underset{\sim}{R}$ matrices presented as Equations (12a) and (13a), becomes

$$
k' = \begin{bmatrix}
m_{11} \\
m_{21} & m_{22} \\
0 & 0 & b_{11} \\
0 & 0 & b_{21} & b_{22} \\
0 & 0 & b_{31} & b_{32} & b_{33} \\
m_{31} & m_{32} & 0 & 0 & 0 & m_{33} \\
m_{41} & m_{42} & 0 & 0 & 0 & m_{43} & m_{44} & & & & & & & \text{SYMMETRIC} \\
0 & 0 & b_{41} & b_{42} & b_{43} & 0 & 0 & b_{44} \\
0 & 0 & b_{51} & b_{52} & b_{53} & 0 & 0 & b_{54} & b_{55} \\
0 & 0 & b_{61} & b_{62} & b_{63} & 0 & 0 & b_{64} & b_{65} & b_{66} \\
m_{51} & m_{52} & 0 & 0 & 0 & m_{53} & m_{54} & 0 & 0 & 0 & m_{55} \\
m_{61} & m_{62} & 0 & 0 & 0 & m_{63} & m_{64} & 0 & 0 & 0 & m_{65} & m_{66} \\
0 & 0 & b_{71} & b_{72} & b_{73} & 0 & 0 & b_{74} & b_{75} & b_{76} & 0 & 0 & b_{77} \\
0 & 0 & b_{81} & b_{82} & b_{83} & 0 & 0 & b_{84} & b_{85} & b_{86} & 0 & 0 & b_{87} & b_{88} \\
0 & 0 & b_{91} & b_{92} & b_{93} & 0 & 0 & b_{94} & b_{95} & b_{96} & 0 & 0 & b_{97} & b_{98} & b_{99} \\
m_{71} & m_{72} & 0 & 0 & 0 & m_{73} & m_{74} & 0 & 0 & 0 & m_{75} & m_{76} & 0 & 0 & 0 & m_{77} \\
m_{81} & m_{82} & 0 & 0 & 0 & m_{83} & m_{84} & 0 & 0 & 0 & m_{85} & m_{86} & 0 & 0 & 0 & m_{87} & m_{88} \\
0 & 0 & b_{10,1} & b_{10,2} & b_{10,3} & 0 & 0 & b_{10,4} & b_{10,5} & b_{10,6} & 0 & 0 & b_{10,7} & b_{10,8} & b_{10,9} & 0 & 0 & b_{10,10} \\
0 & 0 & b_{11,1} & b_{11,2} & b_{11,3} & 0 & 0 & b_{11,4} & b_{11,5} & b_{11,6} & 0 & 0 & b_{11,7} & b_{11,8} & b_{11,9} & 0 & 0 & b_{11,10} & b_{11,11} \\
0 & 0 & b_{12,1} & b_{12,2} & b_{12,3} & 0 & 0 & b_{12,4} & b_{12,5} & b_{12,6} & 0 & 0 & b_{12,7} & b_{12,8} & b_{12,9} & 0 & 0 & b_{12,10} & b_{12,11} & b_{12,12}
\end{bmatrix}
$$

By inspection, the $\underset{\sim}{k}$ stiffness matrix for this element is

$$\underset{\sim}{k} = \begin{bmatrix} m_{33} \\ 0 & b_{44} \\ 0 & b_{54} & b_{55} & & & & & \text{SYMMETRIC} \\ 0 & b_{64} & b_{65} & b_{66} \\ m_{53} & 0 & 0 & 0 & m_{55} \\ m_{63} & 0 & 0 & 0 & m_{65} & m_{66} \\ 0 & b_{74} & b_{75} & b_{76} & 0 & 0 & b_{77} \\ 0 & b_{84} & b_{85} & b_{86} & 0 & 0 & b_{87} & b_{88} \\ 0 & b_{94} & b_{95} & b_{96} & 0 & 0 & b_{97} & b_{98} & b_{99} \\ m_{73} & 0 & 0 & 0 & m_{75} & m_{76} & 0 & 0 & 0 & m_{77} \\ m_{83} & 0 & 0 & 0 & m_{85} & m_{86} & 0 & 0 & 0 & m_{87} & m_{88} \\ 0 & b_{10,4} & b_{10,5} & b_{10,6} & 0 & 0 & b_{10,7} & b_{10,8} & b_{10,9} & 0 & 0 & b_{10,10} \\ 0 & b_{11,4} & b_{11,5} & b_{11,6} & 0 & 0 & b_{11,7} & b_{11,8} & b_{11,9} & 0 & 0 & b_{11,10} & b_{11,11} \\ 0 & b_{12,4} & b_{12,5} & b_{12,6} & 0 & 0 & b_{12,7} & b_{12,8} & b_{12,9} & 0 & 0 & b_{12,10} & b_{12,11} & b_{12,12} \end{bmatrix}$$

Triangular Combined Membrane and Bending Element

Here the $\underset{\sim}{u}$ and $\underset{\sim}{R}$ matrices are defined by Equations (18a) and (19a), respectively.

For this element, the stiffness matrix $\underset{\sim}{k}$ becomes

$$\underset{\sim}{k} = \begin{bmatrix}
m_{11} & & & & & & & & & & & & & & \\
m_{21} & m_{22} & & & & & & & & & & & & & \\
0 & 0 & b_{11} & & & & & & & & & & & & \\
0 & 0 & b_{21} & b_{22} & & & & & & & & & & & \\
0 & 0 & b_{31} & b_{32} & b_{33} & & & & & & & \text{SYMMETRIC} & & & \\
m_{31} & m_{32} & 0 & 0 & 0 & m_{33} & & & & & & & & & \\
m_{41} & m_{42} & 0 & 0 & 0 & m_{43} & m_{44} & & & & & & & & \\
0 & 0 & b_{41} & b_{42} & b_{43} & 0 & 0 & b_{44} & & & & & & & \\
0 & 0 & b_{51} & b_{52} & b_{53} & 0 & 0 & b_{54} & b_{55} & & & & & & \\
0 & 0 & b_{61} & b_{62} & b_{63} & 0 & 0 & b_{64} & b_{65} & b_{66} & & & & & \\
m_{51} & m_{52} & 0 & 0 & 0 & m_{53} & m_{54} & 0 & 0 & 0 & m_{55} & & & & \\
m_{61} & m_{62} & 0 & 0 & 0 & m_{63} & m_{64} & 0 & 0 & 0 & m_{65} & m_{66} & & & \\
0 & 0 & b_{71} & b_{72} & b_{73} & 0 & 0 & b_{74} & b_{75} & b_{76} & 0 & 0 & b_{77} & & \\
0 & 0 & b_{81} & b_{82} & b_{83} & 0 & 0 & b_{84} & b_{85} & b_{86} & 0 & 0 & b_{87} & b_{88} & \\
0 & 0 & b_{91} & b_{92} & b_{93} & 0 & 0 & b_{94} & b_{95} & b_{96} & 0 & 0 & b_{97} & b_{98} & b_{99}
\end{bmatrix}$$

By inspection, the corresponding $\underset{\sim}{k}$ stiffness matrix for the triangular combined membrane and bending element is given by

$$\underset{\sim}{k} = \begin{bmatrix} m_{33} & & & & & & & & \\ 0 & b_{44} & & & & & & & \\ 0 & b_{54} & b_{55} & & & \text{SYMMETRIC} & & & \\ 0 & b_{64} & b_{65} & b_{66} & & & & & \\ m_{53} & 0 & 0 & 0 & m_{55} & & & & \\ m_{63} & 0 & 0 & 0 & m_{65} & m_{66} & & & \\ 0 & b_{74} & b_{75} & b_{76} & 0 & 0 & b_{77} & & \\ 0 & b_{84} & b_{85} & b_{86} & 0 & 0 & b_{87} & b_{88} & \\ 0 & b_{94} & b_{95} & b_{96} & 0 & 0 & b_{97} & b_{98} & b_{99} \end{bmatrix}$$

REFERENCES

1. Thomas, G.O. and Ma, J.H., "User Manual for the Finite Element Program of Structural Analysis," NSRDC Report 2712 (Apr 1968).

2. Paulling, J.R., "The Analysis of Complex Ship Structures by the Finite Element Technique," J. Ship Res., Vol. 8, No. 3, pp 1-14 (Dec 1964).

3. Zienkiewicz, O.C. and Cheung, Y.K., "The Finite Element Method in Structural and Continuum Mechanics," McGraw-Hill, London (1967).

4. Przemieniecki, J.S., "Theory of Matrix Structural Analysis," McGraw-Hill, New York (1968).

5. Melosh, R.J., "A Stiffness Matrix for the Analysis of Thin Plates in Bending," J. Aerospace Sci., Vol. 28, No. 1, pp. 34-42, 64 (Jan 1961).

6. Clough, R.W. and Tocher, J.L., "Finite Element Stiffness Matrices for Analysis of Plate Bending," Proc. Conf. Matrix Methods Struct. Mech., Wright-Patterson Air Force Base, Ohio, October 26-28, 1965; see pp. 515-545 of AFFDL TR 66-80 (1966).

7. Bazeley, G.P., et al., "Triangular Elements in Plate Bending-Conforming and Nonconforming Solutions," Proc. Conf. Matrix Methods Struct. Mech., Wright-Patterson Air Force Base, Ohio, October 26-28, 1965; see pp. 547-576 of AFFDL TR 66-80 (1966).

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13. ABSTRACT

Three additional elements have been added to the element library of the FINEL finite element program: a rectangular membrane element, a rectangular combined membrane and bending element, and a triangular combined membrane and bending element. This report describes the characteristics of the three new elements and includes data on the bending convergence behavior of the combined elements. The stiffness matrices of the three elements are presented in the appendix.

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